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THE STRESS INTENSITY FACTOR HISTORY FOR AN ADVANCING CRACK UNDER THREE-DIMENSIONAL LOADING

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Abstract—The stress intensity factor history due to the uniform extension of a planar crack in an unbounded elastic body under three-dimensional time-independent loading is considered. First, a fundamental (point force) solution is obtained, and this is used to write down the form of the stress intensity factor history for general loading in terms of a superposition integral. A particular traction distribution is also considered.

1. INTRODUCTION

In this article the uniform extension of a half-plane crack in an unbounded elastic body under time-independent loading is considered. This problem is the three-dimensional analogue of the plane strain problem solved by Freund[1]. In Ref. [1] the solution for general loading is obtained by first obtaining a fundamental (concentrated force) solution, and then writing the general solution as a superposition integral.

The fundamental solution to this problem is obtained via the method used by Freund[2] to investigate the impact loading of a half-plane crack. After defining coordinates moving with the crack edge, application of Laplace transforms in time and space results in a certain functional equation which is solved by the Wiener–Hopf technique[3]. This yields a formulation of the field potentials in terms of triple inversion integrals. Attention is then directed towards the stress intensity factor as a function of time and distance along the crack edge, and it is shown that this may be written in terms of a single integral by use of the deHoop method of inversion[4]. A similar analysis to that presented in this paper has been used by Ramirez[5] to investigate the stress intensity factor history due to the motion of point loads on the faces of a half-plane crack.

Once the stress intensity factor history along the crack edge is determined for the fundamental solution, the stress intensity factor for general loading is obtained in integral form by superposition. The paper concludes with the consideration of a particular loading situation.

2. THE FUNDAMENTAL SOLUTION

2.1. *The basic problem*

Consider a half-plane crack moving at speed v in the x-direction, with the crack line oriented parallel to the z-axis in the plane $y = 0$ (see Fig. 1). The location of a point on the crack edge is then given by the position vector $(vt, 0, z)$. At time $t = 0$ a pair of point forces appear at the tip of the crack, one acting on the upper face of the crack at point $(0, 0^+, 0)$ and the other acting on the lower face of the crack at point $(0, 0^-, 0)$. The directions of the forces are along the outward normals to each face, i.e. they tend to open the crack. For

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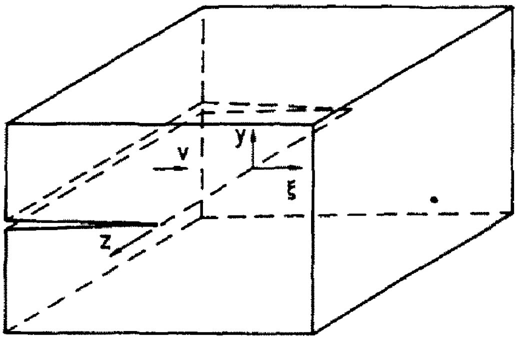

Fig. 1. The geometry of the elastic body.

time $t > 0$ the forces continue to act at the origin, whilst the crack continues to move in the x-direction with speed v. Using symmetry with respect to the plane $y = 0$, only the region $y \geqslant 0$ need be considered. The boundary conditions for displacements (u) and stresses (σ) are

$$\begin{aligned} \sigma_{yy}(x,0,z,t) &= H(t)\,\delta(x)\,\delta(z), \quad x < vt \\ \sigma_{xy}(x,0,z,t) &= \sigma_{yz}(x,0,z,t) = 0, \quad -\infty < x < \infty \\ u_y(x,0,z,t) &= 0, \quad x > vt \end{aligned} \tag{1}$$

with all fields zero for $t < 0$.

The scalar dilatational wave potential ϕ and the vector shear wave potential $\Psi = (\Psi_x, \Psi_y, \Psi_z)$ are now introduced. The potential ϕ satisfies the scalar wave equation

$$\nabla^2\phi - a^2\frac{\partial^2\phi}{\partial t^2} = 0 \tag{2}$$

where a^{-1} is the speed of dilatational waves in the elastic body. The potential Ψ satisfies a similar equation with a replaced by b, where b^{-1} is the shear wave speed. Also, the shear wave potential is divergence free, i.e.

$$\nabla \cdot \Psi = 0. \tag{3}$$

It proves convenient to work with a coordinate system moving with the crack front. If the variable $\xi = x - vt$ is introduced, eqn (1) for ϕ becomes, in the new coordinate system (ξ, y, z)

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + \left(1 - \frac{a^2}{d^2}\right)\frac{\partial^2\phi}{\partial\xi^2} + \frac{2a^2}{d}\frac{\partial^2\phi}{\partial\xi\,\partial t} - a^2\frac{\partial^2\phi}{\partial t^2} = 0 \tag{4}$$

where $d = 1/v$. The shear wave potential Ψ satisfies eqn (4) with a replaced by b.

It is expected that

$$\sigma_{yy}(\xi,0,z,t) \sim \frac{k_1(z,t)}{(2\pi\xi)^{1/2}}, \quad \xi \to 0^+ \tag{5}$$

where k_1 is the stress intensity factor variation along the crack edge, which is to be determined.

2.2. *Solution procedure*

Transform techniques are now used to determine k_1. The following equations are written in terms of the dilatational wave potential ϕ—the shear wave potential may be treated in a similar manner.

First, the one-sided Laplace transform over time is introduced. The parameter is s, and the transformed function is denoted by a superposed hat. Thus

$$\hat{\phi}(\xi, y, z, s) = \int_0^\infty \phi(\xi, y, z, t)\, \mathrm{e}^{-st}\, \mathrm{d}t. \tag{6}$$

For the present, s may be considered to be a real positive parameter.

Next, the dependence on z is suppressed by taking a two-sided Laplace transform with parameter $s\zeta$. The transformed function is denoted by a superposed bar, i.e.

$$\bar{\hat{\phi}}(\xi, y, \zeta, s) = \int_{-\infty}^{\infty} \hat{\phi}(\xi, y, z, s)\, \mathrm{e}^{-s\zeta z}\, \mathrm{d}z. \tag{7}$$

The domain of convergence of this transform is now examined. Noting that the dilatational waves have the greatest speed, the fields will be zero outside of the sphere $x^2+y^2+z^2 = t^2/a^2$. Hence it suffices to examine the convergence of transform (7) when ϕ is given by

$$\phi(\xi, y, z, t) = H[t^2/a^2-(\xi+vt)^2-y^2-z^2] \tag{8}$$

where $H(\cdot)$ is the Heaviside step function.

Consideration of the transform of this function (with ξ, y fixed and $s > 0$) indicates that convergence is satisfied for $-ad/(d^2-a^2)^{1/2} < \mathrm{Re}\,\zeta < ad/(d^2-a^2)^{1/2}$. For the present it proves convenient to consider ζ as a real parameter within this range—the transform may be analytically continued into its region of convergence in the complex ζ-plane when necessary.

Finally, a two-sided Laplace transform over ξ is taken. The transformed function is denoted by a capital letter, and the transform parameter is $s\eta$, i.e.

$$\Phi(\eta, y, \zeta, s) = \int_{-\infty}^{\infty} \bar{\hat{\phi}}(\xi, y, \zeta, s)\, \mathrm{e}^{-s\eta\xi}\, \mathrm{d}\xi. \tag{9}$$

The domain of convergence of this transform is investigated by using the step function defined in eqn (8) coupled with the range of ζ given in the above paragraph. It is found that, for convergence, η must lie in the strip

$$-a_2 < \mathrm{Re}\,(\eta) < a_1 \tag{10}$$

where

$$a_{2,1} = \frac{\pm a^2 d + d[a^4+(a^2-\zeta^2)(d^2-a^2)]^{1/2}}{d^2-a^2} \tag{11}$$

and

$$-a_0 < \zeta < a_0; \quad a_0 = ad/(d^2-a^2)^{1/2}. \tag{12}$$

The governing equations for the potentials ϕ and Ψ are now transformed, leading to the ordinary differential equations

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2} - s^2\alpha^2\Phi = 0$$
$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} - s^2\beta^2\Psi = 0 \tag{13}$$

where

$$\alpha^2 = a^2\left(1-\frac{\eta}{d}\right)^2 - \eta^2 - \zeta^2$$
$$\beta^2 = b^2\left(1-\frac{\eta}{d}\right)^2 - \eta^2 - \zeta^2. \tag{14}$$

The solutions to eqns (13) may be written in the form

$$\Phi(\eta, y, \zeta, s) = \frac{A(\eta,\zeta,s)}{s^3}\,\mathrm{e}^{-\alpha s y}$$
$$\Psi(\eta, y, \zeta, s) = \frac{\mathbf{B}(\eta,\zeta,s)}{s^3}\,\mathrm{e}^{-\beta s y} \tag{15}$$

where the forms of the constant terms are introduced for convenience—it turns out that A and $\mathbf{B}$ are independent of s.

The boundary conditions are now transformed. With reference to eqns (1), define functions σ_+ and u_- by

$$\sigma_+(\xi, z, t) = \sigma_{yy}(\xi, 0, z, t), \quad \xi > 0$$
$$u_-(\xi, z, t) = u_y(\xi, 0, z, t), \quad \xi < 0. \tag{16}$$

The boundary conditions then transform to

$$\Sigma_{yy}(\eta, 0, \zeta, s) = \frac{1}{s}\left[\Sigma_+(\eta,\zeta,s) - \frac{d}{\mu(\eta-d)}\right] \tag{17}$$

$$\Sigma_{xy}(\eta, 0, \zeta, s) = 0 \tag{18}$$

$$U_y(\eta, 0, \zeta, s) = \frac{1}{s^2}U_-(\eta,\zeta,s) \tag{19}$$

with the functions U_- and Σ_+ defined by

$$\Sigma_+(\eta,\zeta,s) = s\int_0^\infty \bar{\bar{\sigma}}_+(\xi,\zeta,s)\,\mathrm{e}^{-s\eta\xi}\,\mathrm{d}\xi$$

and

$$U_-(\eta,\zeta,s) = s^2\int_{-\infty}^0 \bar{\bar{u}}_-(\xi,\zeta,s)\,\mathrm{e}^{-s\eta\xi}\,\mathrm{d}\xi. \tag{20}$$

It should be noted that Σ_+ is an analytic function of η for Re $(\eta) > -a_2$, and U_- is an analytic function of η for Re $(\eta) < a_1$, where $a_{2,1}$ are defined in eqn (11). From now on, a plus sign denotes a function analytic for Re $(\eta) > -a_2$, and a minus sign a function analytic for Re $(\eta) < a_1$.

Equation (17) may be manipulated into the form

$$\Sigma_{yy}(\eta,0,\zeta,s) = \frac{H_+(\eta,\zeta,s)}{s(\eta-d)} \tag{21}$$

with

$$H_+(d,\zeta,s) = -d. \tag{22}$$

Using the relationships of potentials to stresses and displacements[6], the transformed boundary conditions (18), (19), (21) and eqns (3) and (15), the following system of equations relating A, $\mathbf{B}$, H_+ and U_- is obtained:

$$\begin{aligned} [b^2(1-\eta/d)^2-2\eta^2-2\zeta^2]A-2\beta\zeta B_x+2\beta\eta B_z &= H_+/\mu(\eta-d) \\ 2\eta\alpha A+(\beta^2-\zeta^2)B_x+\eta\beta B_y+\eta\zeta B_z &= 0 \\ 2\zeta\alpha A-\eta\zeta B_x-\zeta\beta B_y+(\eta^2-\zeta^2)B_z &= 0 \\ \alpha A-\zeta B_x+\eta B_z &= U_- \\ \eta B_x-\beta B_y+\zeta B_z &= 0. \end{aligned} \tag{23}$$

It is noted that s does not appear in the above equations, and hence A, $\mathbf{B}$, H_+ and U_- are functions of η and ζ only. Reduction of the system of eqns (23) shows that U_- and H_+ are related by

$$\frac{H_+(\eta,\zeta)}{U_-(\eta,\zeta)} = -\frac{\mu d^2}{b^2}\frac{R(\eta,\zeta)}{(\eta-d)\alpha(\eta,\zeta)} \tag{24}$$

with

$$R(\eta,\zeta) = 4\alpha(\eta,\zeta)\beta(\eta,\zeta)(\eta^2+\zeta^2)+[b^2(1-\eta/d)^2-2\eta^2-2\zeta^2]^2. \tag{25}$$

Equation (24) is now solved for H_+ and U_- by use of the Wiener–Hopf technique[3]. The first step is to manipulate the equation into a relationship between a plus function and a minus function.

The function R may be written in the form

$$R(\eta,\zeta) = (\eta^2+\zeta^2)^2\bar{R}(\eta,\zeta) \tag{26}$$

where

$$\bar{R}(\eta,\zeta) = [2-b^2/\lambda^2]^2-4[1-a^2/\lambda^2]^{1/2}[1-b^2/\lambda^2]^{1/2} \tag{27}$$

with

$$\lambda^2 = (\eta^2+\zeta^2)/(1-\eta/d)^2. \tag{28}$$

Equation (27) is the Rayleigh wave function[6] (in λ), and has two, and only two, roots $\lambda = \pm c$, where c is the inverse of the Rayleigh wave speed. Hence $R(\eta,\zeta)$ has roots in the η-plane which are solutions of

$$(\eta^2+\zeta^2)/(1-\eta/d)^2 = c^2. \tag{29}$$

These roots are

$$\eta = c_1, -c_2 \tag{30}$$

where

$$c_{2,1}(\zeta) = \frac{\pm c^2 d + d[c^4 + (c^2-\zeta^2)(d^2-c^2)]^{1/2}}{d^2-c^2}. \tag{31}$$

By inspection it may be seen that $R(\eta,\zeta)$ also has a zero at $\eta = d$. To find the order of this zero, it is noted that $\eta = d$ corresponds to $\lambda \to \infty$ in eqn (27). A simple expansion shows that $\bar{R} = O(1/\lambda^2)$, $\lambda \to \infty$, and this, coupled with eqn (28) shows that R has a double zero at $\eta = d$. Hence R has four, and only four, zeros in the complex η-plane, these being the real zeros $\eta = d, d, c_1, -c_2$.

Consider now a function S defined by

$$S(\eta,\zeta) = -\frac{R(\eta,\zeta)}{k(\eta-c_1)(\eta+c_2)(\eta-d)^2} \tag{32}$$

where

$$k = -\lim_{\eta\to\infty}\frac{R(\eta,\zeta)}{\eta^4} = 4(1-a^2/d^2)^{1/2}(1-b^2/d^2)^{1/2}-(2-b^2/d^2)^2. \tag{33}$$

The function S has no poles or zeros in the η-plane, the only singularities being the branch points of the functions α and β. Note that the branch cuts of all functions considered here are taken to lie on the real axis outside of the strip of analyticity, $-a_2 < \mathrm{Re}(\eta) < a_1$, of the transforms. The function s may be written as the product of a plus function and a minus function via use of Cauchy's integral theorem[7]. The result is

$$S(\eta,\zeta) = S_+(\eta,\zeta)S_-(\eta,\zeta) \tag{34}$$

where

$$S_\pm(\eta,\zeta) = \exp\left(-\frac{1}{\pi}\int_{a_{2,1}}^{b_{2,1}}\tan^{-1}\left[\frac{4|\alpha||\beta|(v^2+\zeta^2)}{\{b^2(1\pm v/d)^2-2v^2-2\zeta^2\}^2}\right]\frac{dv}{v\pm\eta}\right) \tag{35}$$

with

$$b_{2,1} = \frac{\pm b^2 d + d[b^4+(b^2-\zeta^2)(d^2-b^2)]^{1/2}}{d^2-b^2}. \tag{36}$$

Note that $a_{2,1}$ and $b_{2,1}$ are the branch points of functions α and β, respectively.

With reference to eqn (24), it remains to split the function α. This may be achieved by writing

$$\alpha = k_a\alpha_+\alpha_- \tag{37}$$

with

$$k_a = (1-a^2/d^2)^{1/2} \tag{38}$$

and

$$\begin{aligned} \alpha_+(\eta,\zeta) &= [a_2(\zeta)+\eta]^{1/2} \\ \alpha_-(\eta,\zeta) &= [a_1(\zeta)-\eta]^{1/2}. \end{aligned} \tag{39}$$

By use of eqns (34) and (37), eqn (24) may be manipulated into the form

$$\frac{H_+(\eta,\zeta)\alpha_+(\eta,\zeta)}{S_+(\eta,\zeta)[\eta+c_2(\zeta)]} = \mu\frac{d^2}{b^2}\frac{k}{k_a}\frac{U_-(\eta,\zeta)}{\alpha_-(\eta,\zeta)}S_-(\eta,\zeta)[\eta-c_1(\zeta)](\eta-d). \tag{40}$$

The right-hand side of the above equation is analytic for $\mathrm{Re}(\eta) < a_1$, the left-hand side is analytic for $\mathrm{Re}(\eta) > -a_2$, and hence, by analytic continuation, each side of eqn (40) represents the same entire function. Use of continuity of displacement at the crack tip and the asymptotic form (4) of σ_{yy} shows that each side of eqn (40) tends to a constant as $\eta \to \infty$ in the respective half planes. Hence, by Liouville's theorem, the entire function represented by each side of eqn (40) is a constant, and use of eqn (22) then gives

$$H_+(\eta,\zeta) = -d\left[\frac{d+a_2(\zeta)}{\eta+a_2(\zeta)}\right]^{1/2}\left[\frac{\eta+c_2(\zeta)}{d+c_2(\zeta)}\right]\frac{S_+(\eta,\zeta)}{S_+(d,\zeta)}. \tag{41}$$

The function U_- may be obtained by replacing H_+ in eqn (40) by the above result.

Having solved the functional eqn (40), the displacement potentials may be written down in the form of triple inversion integrals. However, it turns out that the stress intensity factor $k_1(z,t)$ may be obtained in the form of a single real integral.

2.3. *Determination of the function* $k_1(z,t)$

The analysis in this section follows closely that used by Freund[2]. With reference to eqn (3), the transformed stress intensity factor may be written as

$$\bar{\hat{k}}_1(\zeta,s) = \lim_{\xi\to 0^+}[(2\pi\xi)^{1/2}\bar{\hat{\sigma}}_{yy}(\xi,0,\zeta,s)]. \tag{42}$$

Use of Abels' theorem on the asymptotic properties of transforms[7] gives

$$\bar{\hat{k}}_1(\zeta,s) = \lim_{\eta\to\infty}[(2s\eta)^{1/2}\Sigma_{yy}(\eta,0,\zeta,s)] \tag{43}$$

and this result, coupled with eqns (21) and (41), yields

$$\hat{k}_1(\zeta,s) = d\left(\frac{2}{s}\right)^{1/2}Q(\zeta) \tag{44}$$

with

$$Q(\zeta) = \frac{[d+a_2(\zeta)]^{1/2}}{d+c_2(\zeta)}\frac{1}{S_+(d,\zeta)}. \tag{45}$$

After relaxing the constraint that ζ is real, inversion of the Laplace transform over z gives

$$\hat{k}_1(z,s) = \frac{d}{\pi i}\left(\frac{s}{2}\right)^{1/2}\int_{q-i\infty}^{q+i\infty}Q(\zeta)\,e^{zs\zeta}\,d\zeta \tag{46}$$

where

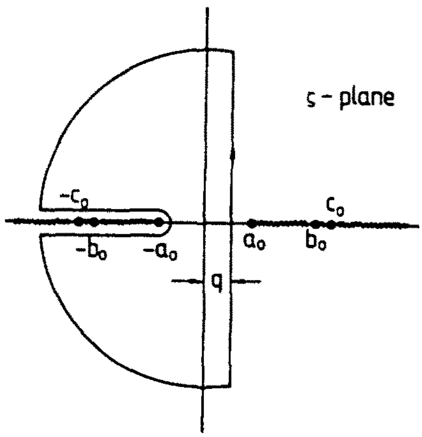

Fig. 2. Deformation of the inversion path in the complex ζ-plane.

$$-a_0 < q < a_0 \tag{47}$$

with a_0 defined in eqn (12).

It is desired to write eqn (46) in such a way that the inversion of the one-sided Laplace transform over time may be obtained via a convolution integral. The function $Q(\zeta) = O(\zeta^{-1/2})$, $\zeta \to \infty$, and Jordan's lemma may be used[7] to deform the integral in eqn (46) onto the branch cut from $-a_0$ to $-\infty$, if $z > 0$ is assumed for the present. On noting that $\bar{Q}(\zeta) = Q(\bar{\zeta})$ and that Q is an even function of ζ, $\hat{k}_1$ may be written in the form

$$\hat{k}_1(z,s) = \frac{-d}{\pi}\frac{(2s)^{1/2}}{z}\int_{a_0 z}^{\infty} \mathrm{Im}\,[Q(u/z+\mathrm{i}0)]\,\mathrm{e}^{-su}\,\mathrm{d}u \tag{48}$$

where the i0 part in the argument of function Q indicates evaluation on the upper side as the branch cut from $-a_0$ to $-\infty$ (see Fig. 2).

In order that the convolution theorem for Laplace transforms may be applied, $\hat{k}_1$ must be written as the product of two such transforms. The integral term in eqn (48) is itself a Laplace transform, but the $s^{1/2}$ term multiplying it is of too high an order to be such a transform. To rectify this, a function H is defined by

$$k_1(z,t) = \frac{\partial H(z,t)}{\partial t}; \quad H(z,0) = 0. \tag{49}$$

Then

$$\hat{k}_1(z,s) = s\hat{H}(z,s) \tag{50}$$

which, coupled with eqn (48), yields

$$\hat{H}(z,s) = \frac{d}{z\pi}\left(\frac{2}{s}\right)^{1/2}\int_{a_0 z}^{\infty} \mathrm{Im}\,[Q(u/z+\mathrm{i}0)]\,\mathrm{e}^{-su}\,\mathrm{d}u. \tag{51}$$

The integral term in the expression for $\hat{H}$ is the Laplace transform of $Q(u/z+\mathrm{i}0)H(u-a_0 z)$ and the $s^{-1/2}$ term is the Laplace transform of $(\pi t)^{-1/2}$. It follows, by convolution, that, for $z > 0$

$$H(z,t) = \frac{d}{z\pi}\left(\frac{2}{\pi}\right)^{1/2}\int_{a_0 z}^{t} \mathrm{Im}\,\frac{[Q(u/z+\mathrm{i}0)]}{(t-u)^{1/2}}\,\mathrm{d}u, \quad t > a_0 z$$
$$= 0, \quad t < a_0 z. \tag{52}$$

Upon noting that $k_1(z,t)$ is an even function of z, use of eqns (49) and (52) gives

$$k_1(z,t) = \frac{d}{\pi|z|}\left(\frac{2}{\pi}\right)^{1/2}\frac{\partial}{\partial t}\int_{a_0|z|}^{t} \mathrm{Im}\,\frac{[Q(u/|z|+\mathrm{i}0)]}{(t-u)^{1/2}}\,\mathrm{d}u, \quad |z| < t/a_0$$
$$= 0, \quad |z| > t/a_0. \tag{53}$$

Integration by parts gives, finally

$$k_1(z,t) = \frac{2^{1/2}}{(\pi|z|)^{3/2}}k(\tau) \quad (\tau > 1)$$
$$= 0 \qquad (\tau < 1) \tag{54}$$

where $k(\tau)$ is the dimensionless integral

$$k(\tau) = da_0^{1/2}\int_1^{\tau} \mathrm{Im}\,\frac{[Q'(va_0+\mathrm{i}0)]}{(\tau-v)^{1/2}}\,\mathrm{d}v \tag{55}$$

with

$$\tau = t/a_0|z|. \tag{56}$$

It is noted that the parameter a_0 defined in eqn (12) is such that a_0^{-1} is the apparent speed at which dilatational waves move along the crack edge. Hence, as indicated by eqns (54), the first dilatational wave reaches a fixed point on the crack line at time $t = a_0|z|$. Similarly, the quantities

$$b_0 = bd/(d^2-b^2)^{1/2}; \quad c_0 = cd/(d^2-c^2)^{1/2} \tag{57}$$

are such that b_0^{-1} and c_0^{-1} are the speeds at which shear waves and Rayleigh waves move along the crack edge, respectively—note that $a_0 < b_0 < c_0$.

2.4. *Properties of* $k_1(z,t)$

One interesting feature of $k_1(z,t)$ is that it is discontinuous at the front of the pulse generated by the point loads. The jump may be written as

$$\Delta k_1 = \lim_{t\to a_0|z|^+}[k_1(z,t)] \tag{58}$$

and it may be shown, by a simple asymptotic analysis of the integral in eqn (55), that

$$\Delta k_1 = -\frac{d^2-c^2}{1+\left(\dfrac{c^2-a^2}{d^2-a^2}\right)^{1/2}}\left(\frac{a}{\pi|z|}\right)^{1/2}\frac{1}{2(d^2-a^2)^{1/4}d^2|z|}\,\frac{1}{S_+(d,a_0)}. \tag{59}$$

The asymptotic form of $k_1(z,t)$ for large t at a fixed point z is also of interest. With reference to eqns (54), this asymptotic result may be obtained by fixing z and letting $t\to\infty$. This limit is equivalent to holding t fixed and letting $z\to 0$, and hence, as $k_1(0,t)$ is expected to be finite, the function $k(\tau)$ must have the property that

$$k(\tau) \sim p\tau^{-3/2}, \quad \tau \to \infty \tag{60}$$

where p is a constant. Consequently

$$k_1(z,t) \sim \left(\frac{2a_0}{\pi}\right)^{1/2} \frac{a_0 p}{\pi} t^{-3/2}, \quad t/|z| \to \infty. \tag{61}$$

The constant p is found in the Appendix by an asymptotic analysis of the integral defining $k(\tau)$.

One useful check on the result obtained for $k_1(z,t)$ is that it should reduce to the stress intensity factor history for the corresponding two-dimensional line load problem solved by Freund[1] when integrated over the range $-\infty < z < \infty$. If the integration is performed on eqn (46), and the transform over time inverted, the result is

$$\int_{-\infty}^{\infty} k_1(z,t)\,\mathrm{d}z = \left(\frac{2}{\pi t}\right)^{1/2} \frac{d-c}{(d-a)^{1/2} S_+(d,0)} \tag{62a}$$

which agrees with the result presented by Freund[1].

Another check on result (54) is that if the wave slownesses are nominally set equal to zero in eqns (54), $k_1(z,t)$ should reduce to the corresponding stress intensity factor for the static problem of a pair of unit normal line loads applied to the faces of a half plane crack at a distance t/d from the crack edge[8]. Letting a, b, $c \to 0$ in eqns (54) gives

$$k_1(z,t) = \frac{1}{2}\left(\frac{2d}{\pi t}\right)^{3/2} \frac{1}{1+(\mathrm{d}z/t)^2} \tag{62b}$$

which agrees with the static result given in Ref. [8].

A numerical integration of the non-dimensional function $k(\tau)$ is now carried out. Before continuing, the function $S_+(d,\zeta)$ is written in a more convenient form for computations. Using eqns (26)–(28) for motivation, the substitution

$$w^2 = (v^2+\xi^2)/(1-v/d)^2 \tag{63}$$

is made in the integral part of the function S_+ defined in eqn (35).

This leads to

$$S_+(d,\zeta) = \exp\left\{-\frac{1}{\pi}\int_a^b \tan^{-1}\left[\frac{w^2(w^2-a^2)^{1/2}(b^2-w^2)^{1/2}}{(b^2/2-w^2)^2}\right] f(w,\zeta)\,\mathrm{d}w\right\} \tag{64}$$

where

$$f(w,\zeta) = \frac{d^2 e(w,\zeta) + (d^2+\zeta^2)(d^2+w^2) - 2d^2\zeta^2}{[d^2+e(w,\zeta)](d^2-w^2)} \cdot \frac{w}{e(w,\zeta)} \tag{65}$$

with

$$e(w,\zeta) = [w^2(d^2+\zeta^2) - \zeta^2 d^2]^{1/2}. \tag{66}$$

Consider now the non-dimensional function $k(\tau)$ defined by eqn (55). It can be seen that $k(\tau) = k(b/a, c/a, c/d, t)$ and hence, if the ratios b/a and c/a are fixed, the variation of k with τ is a function of c/d only.

In Fig. 3 the function k is plotted against τ for various values of c/d when Poisson's ratio $\nu = 0.3$ ($b = 1.87a$, $c = 2.02a$). The graphs show that the stress intensity factor at a fixed point z starts at a negative value upon arrival of the first dilatational wave at time $t = a_0|z|$. There is a kink in each curve corresponding to the arrival of the first shear wave at time $t = b_0|z|$, and a logarithmic singularity when the first Rayleigh wave arrives at time $t = c_0|z|$. After this time the stress intensity factor stays positive and decays like $t^{-3/2}$ as $t \rightarrow \infty$.

3. GENERAL LOADING

The arguments used in this section are similar to those used by Freund[1] for the mode I plane strain problem. The fundamental solution derived in Section 2 is used to write the solution for general loading as a superposition integral.

The crack is considered to be stationary for time $t < 0$ under equilibrium loading, with the normal stress on the half-plane $y = 0$ ahead of the crack being given by $\sigma_{yy} = -p(x, z)$, with the minus sign introduced for convenience. At time $t = 0$, the crack begins to move with speed v in the positive x-direction, creating new stress-free surfaces ($0 < x < vt$, $y = 0^{\mp}$, $-\infty < z < \infty$). The resulting field may be considered to be the superposition of a dynamic field created by imposing tractions $p(x, z)$ ($0 < x < vt$, $-\infty < z < \infty$) on the crack faces, and a static field corresponding to the equilibrium field. The solution to the dynamic problem is now obtained in integral form.

Consider the point-load problem of Section 2, with the exception that point loads of strength $p(x', z')\,dx'\,dz'$ pass through the edge of the crack at the point $x = x'$, $z = z'$ at time $t = t' = x'/v$. The corresponding stress intensity factor is given, as a function of z and t, by $k_1(z-z', t-x'/v)p(x', z')\,dx'\,dz'$. The full dynamic stress intensity factor K_1 corresponding to the applied traction $p(x', z')$ may consequently be written as

$$K_1(z, t) = \int_{-\infty}^{\infty} \int_0^{vt} k_1(z-z', t-x'/v)p(x', z')\,dx'\,dz'. \quad (67)$$

This formulation of K_1 is useful for numerical computations involving a given traction distribution. However, if further analytic manipulation of K_1 for a specific traction distribution is desired, it may be better to work with eqn (46) and build up an alternative integral form for K_1. This is made clear in the next section, where a particular loading situation is considered.

3.1. *A particular traction distribution*

The traction distribution considered is the constant rectangular distribution

$$p(x, z) = p_0; \quad 0 < x < vt, \quad -z_0 < z < z_0. \quad (68)$$

The solution to this problem may be obtained via the superposition of concentrated line loads, i.e.

$$K_1(z, t) = p_0 \int_0^{vt} k_1^*(z, t-x'/v)\,dx' \quad (69)$$

where $k_1^*(z, t)$ is the stress intensity factor for a unit line load of finite length ($-z_0 < z < z_0$) passing through the crack tip at time $t = 0$.

The first step in the procedure is to investigate k_1^*, which may be written as

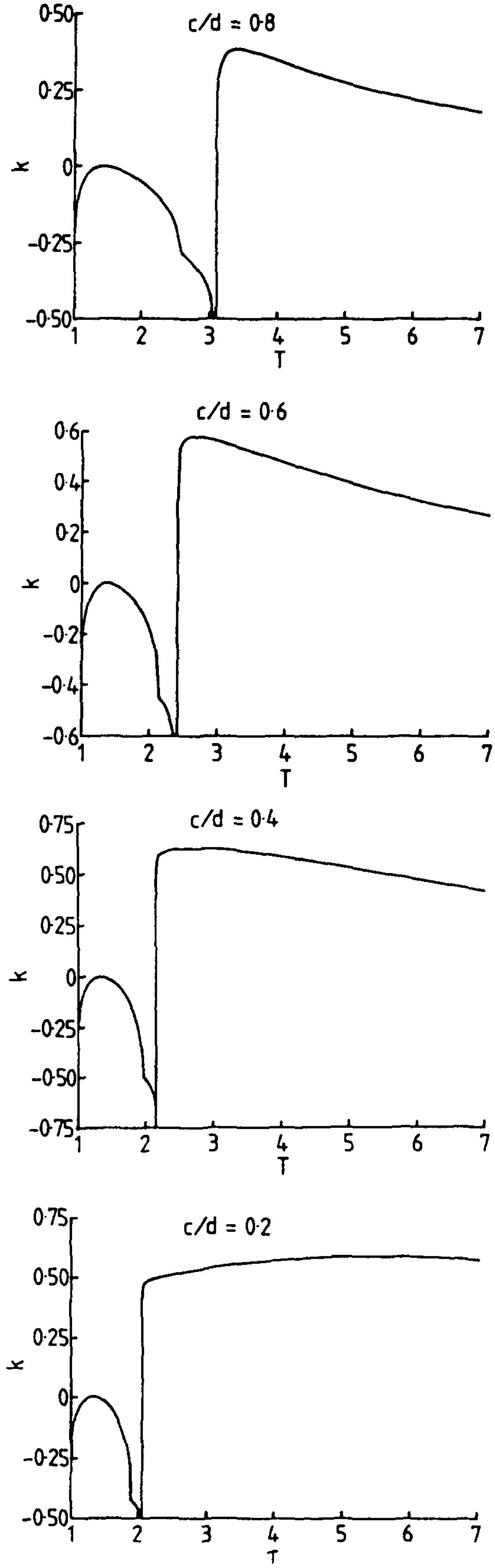

Fig. 3. The normalized stress intensity factor history $\bar{k} = (\pi|z|)^{3/2}k_1/\sqrt{2}$ vs $\tau = t/a_0|z|$ for the fundamental solution.

$$k_{\mathrm{I}}^{*}(z,t) = \int_{-z_0}^{z_0} k_{\mathrm{I}}(z-z',t)\,\mathrm{d}z'. \tag{70}$$

Using eqn (46), eqn (70) becomes

$$\hat{k}_{\mathrm{I}}^{*}(z,s) = \frac{d}{\pi \mathrm{i}}\left(\frac{s}{2}\right)^{1/2} I(s,z) \tag{71}$$

where

$$I(s,z) = \int_{-z_0}^{z_0} \mathrm{d}z' \int_{q-\mathrm{i}\infty}^{q+\mathrm{i}\infty} Q(\zeta)\,\mathrm{e}^{s(z-z')\zeta}\,\mathrm{d}\zeta \tag{72}$$

and $0 < q < a_0$.

Interchanging the order of integration, and evaluating the integral over z', leads to

$$I(z,s) = \frac{1}{s}[J(z+z_0,s) - J(z-z_0,s)] \tag{73}$$

where

$$J(x,s) = \int_{q-\mathrm{i}\infty}^{q+\mathrm{i}\infty} \frac{Q(\zeta)}{\zeta}\,\mathrm{e}^{sx\zeta}\,\mathrm{d}\zeta. \tag{74}$$

For simplicity, $z > 0$ is assumed for the present. Upon noting that $\overline{Q(\zeta)} = Q(\bar{\zeta})$, and also that $Q(\zeta) = Q(-\zeta)$, the integral J may be manipulated into the form

$$\begin{aligned} J(x) &= 2\pi\mathrm{i}\; Q(0) - 2\mathrm{i}\; J_1(x,s), \quad x > 0 \\ &= 2\mathrm{i}\; J_1(-x,s), \quad x < 0 \end{aligned} \tag{75}$$

with

$$J_1(x,s) = \int_{a_0}^{\infty} \mathrm{Im}\left[\frac{Q(u+\mathrm{i}0)}{u}\right]\mathrm{e}^{-sxu}\,\mathrm{d}u. \tag{76}$$

Consequently, with reference to eqns (71) and (73), the time transform of k_{I}^{*} is given by

$$\begin{aligned} \hat{k}_{\mathrm{I}}^{*}(z,s) &= \frac{d}{\pi}\left(\frac{2}{s}\right)^{1/2}\{\pi Q(0) - J_1(z_0-z) - J_1(z+z_0)\}, \quad z < z_0 \\ &= \frac{d}{\pi}\left(\frac{2}{s}\right)^{1/2}\{J_1(z-z_0) - J_1(z+z_0)\}, \quad z > z_0. \end{aligned} \tag{77}$$

It is noted that there is an apparent jump in $\hat{k}_{\mathrm{I}}^{*}$ at $z = z_0$ given by

$$\Delta\hat{k}_{\mathrm{I}}^{*}(z_0,s) = 4\mathrm{i}\int_{a_0}^{\infty} \mathrm{Im}\left[\frac{Q(u+\mathrm{i}0)}{u}\right]\mathrm{d}u - 2\pi\mathrm{i}\; Q(0) \tag{78}$$

but consideration of the integral $J(0,s)$ defined in eqn (74) shows that $\Delta\hat{k}_{\mathrm{I}} = 0$.

Inspection of the form of the integral J_1 defined in eqn (76) shows that it is of the form for which the convolution theorem for Laplace transforms may be applied to eqns (77). This results in the following integral form for k_I^* (with $z > 0$):

$$
\begin{aligned}
k_I^*(z,t) &= \frac{d}{\pi}\left(\frac{2}{\pi}\right)^{1/2}[g(z-z_0,t)-g(z+z_0,t)], \quad |z| > z_0 \\
&= \frac{d}{\pi}\left(\frac{2}{\pi}\right)^{1/2}[\pi Q(0)t^{-1/2}-g(z-z_0,t)-g(z+z_0,t)], \quad z < z_0
\end{aligned}
\tag{79}
$$

with

$$
g(x,t) = \int_{a_0 x}^{t} \operatorname{Im}\left[\frac{Q(u/x+i0)}{u}\right]\frac{du}{(t-u)^{1/2}}, \quad x > 0. \tag{80}
$$

The stress intensity factor for the rectangular distribution may be found from eqn (69). The result, after some algebraic manipulation, is

$$
K_I(z,t) = \frac{p_0}{\pi}\left(\frac{2}{\pi}\right)^{1/2}[\pi t^{1/2}Q(0) - J_2(z-z_0,t) - J_2(z+z_0,t)], \quad z < z_0 \tag{81a}
$$

$$
= \frac{p_0}{\pi}\left(\frac{2}{\pi}\right)^{1/2}[J_2(z-z_0,t) - J_2(z+z_0,t)], \quad z > z_0 \tag{81b}
$$

where

$$
J_2(x,t) = 2x^{1/2}\int_{a_0}^{t/x} \operatorname{Im}\left[\frac{Q(w+i0)}{w}\right](t/x-w)^{1/2}\,dw. \tag{82}
$$

Note that $z > 0$ has been assumed. The results for $z < 0$ may be written down after noting that K_I is an even function of z.

Consider $K_I(z,t)$ in the range $-z_0 < z < z_0$. The first term in eqn (81a) is the stress intensity factor history for a distribution of strength p_0 with infinite extent in the z-direction—this may be obtained by a superposition integral using the line load solution obtained by Freund[1]. The second and third terms are corrections which take into account the finiteness of the traction distribution in the z-direction—they may be considered to correspond to waves emanating from the boundaries $z = \pm z_0$ of the traction distribution on the crack edge. The contribution to K_I from these terms tends to zero for large values of z_0. For $z > z_0$, K_I may be thought of as the superposition of two waves centred at the points $z = \pm z_0$ on the crack line.

With reference to eqn (82), a non-dimensional integral $J_3(\lambda)$ is defined by

$$
J_3(\lambda) = \frac{J_2(x,t)}{2x^{1/2}}, \quad \lambda = t/a_0 x. \tag{83}
$$

In Fig. 4 the integral J_3 is plotted against λ for various values of c/d when Poisson's ratio $\nu = 0.3$ ($b = 1.87a$, $c = 2.02a$). Note that, unlike the results from Section 2, there is no discontinuity in the stress intensity factor when the first dilatational wave arrives at a fixed point z on the crack edge. Also, the stress intensity factor does not become singular when the first Rayleigh wave reaches the fixed point z.

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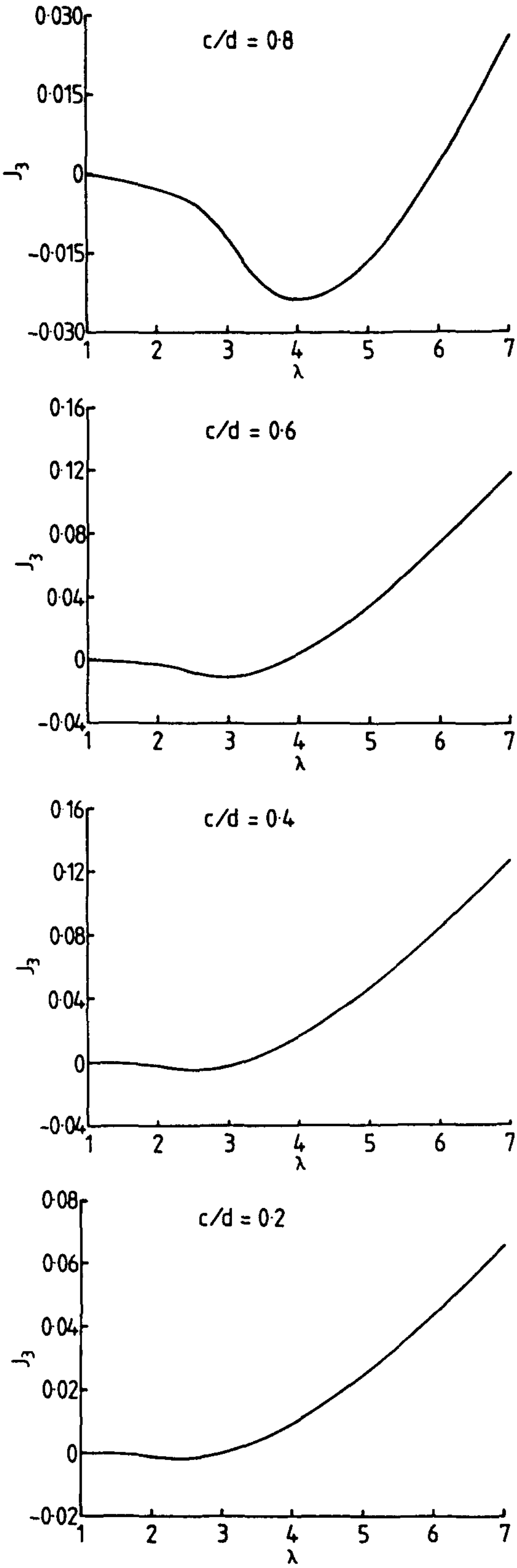

Fig. 4. The dimensionless integral $J_3(\lambda)$, defined in eqn (83), vs λ.

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APPENDIX

The asymptotic form for large τ of the function $k(\tau)$, as defined in eqn (55), is considered. First, $k(\tau)$ is written in the form

$$k(\tau) = \frac{d}{a_0}\frac{\partial I}{\partial \tau} \tag{A1}$$

with

$$I(\tau) = \int_{a_0}^{\tau a_0} \frac{\operatorname{Im}\,[Q(u+\mathrm{i}0)]}{(\tau a_0 - u)^{1/2}}\,\mathrm{d}u \tag{A2}$$

where Q is defined in eqn (45).

A simple asymptotic analysis of Q shows that

$$\operatorname{Im}\,[Q(u+\mathrm{i}0)] \sim A(u-a_0)^{-1/2} + O(u^{-3/2}), \quad u \to \infty \tag{A3}$$

with constant A being given by

$$A = \frac{(d^2-c^2)^{1/2}}{(2d)^{1/2}(d^2-a^2)^{1/4}} S_+(d, -\infty) \tag{A4}$$

where

$$S_+(d, -\infty) = \exp\left\{-\frac{1}{\pi}\int_a^b \tan^{-1}\left[\frac{w^2(w^2-a^2)^{1/2}(b^2-w^2)^{1/2}}{(b^2/2-w^2)^2}\right]\frac{w\,\mathrm{d}w}{d^2-w^2}\right\}. \tag{A5}$$

The integral $I(\tau)$ defined in line (A2) may be written in the form

$$I(\tau) = \int_{a_0}^{\tau a_0}\left\{\operatorname{Im}\,[Q(u+\mathrm{i}0)] - \frac{A}{(u-a_0)^{1/2}}\right\}\frac{\mathrm{d}u}{(\tau a_0 - u)^{1/2}} + A\int_{a_0}^{\tau a_0}\frac{\mathrm{d}u}{(\tau a_0-u)^{1/2}(u-a_0)^{1/2}}. \tag{A6}$$

Letting $\tau \to \infty$ results in

$$I(\tau) \sim \pi A + (a_0\tau)^{-1/2}\int_{a_0}^{\infty}\left\{\operatorname{Im}\,[Q(u+\mathrm{i}0)] - \frac{A}{(u-a_0)^{1/2}}\right\}\mathrm{d}u. \tag{A7}$$

Finally, from eqn (A1), $k(\tau)$ satisfies

$$k(\tau) \sim P\tau^{-3/2}, \quad \tau \to \infty \tag{A8}$$

with constant P being given by

$$P = \frac{d}{2a_0^{3/2}}\int_{a_0}^{\infty}\left\{\frac{A}{(u-a_0)^{1/2}} - \operatorname{Im}\,[Q(u+\mathrm{i}0)]\right\}\mathrm{d}u. \tag{A9}$$